The Ohio State University

BLACK MATH HISTORY
HIDDEN FIGURES REVEALED

Sum of Infinite Geometric Series Activity

Grade or age level: Grades 2-7 **Time:** 60 minutes **Form of work:** Individual

Background

Rada Higgins got her PhD in mathematics from the Ohio State University in 1974. Her dissertation focused on the limits, or end behaviors, of series and sequences. Dr. Higgins has also been interested in creating materials for students to learn mathematics and other subjects at home. The following lesson plan is appropriate for students to learn about sums of infinite geometric series in simple terms, much like how Dr. Higgins would have wanted.

Geometric Sequence

A geometric sequence is a sequence with a constant growing factor. For example,

1, 2, 4, 8, 16, 32, 64, 128, …

depicts a geometric sequence with a growing factor of 2. We begin with 1 as the first term and obtain the second term by multiplying the first times 2. Thus, the second term is 2, since $1 \times 2 = 2$. For the third term, we now multiply the second term, in this case 2, by our constant 2, to get 4. This pattern continues forever, each term doubling the previous one.

$$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} 32, \ldots$$

The constant we multiply by does not have to be a whole number. In fact, the sequences we will explore in this lesson have a growth constant between 0 and 1. For example, with ½ the sequence would be

$$1 \xrightarrow{\times \frac{1}{2}} \frac{1}{2} \xrightarrow{\times \frac{1}{2}} \frac{1}{4} \xrightarrow{\times \frac{1}{2}} \frac{1}{8} \xrightarrow{\times \frac{1}{2}} \frac{1}{16} \xrightarrow{\times \frac{1}{2}} \frac{1}{32}, \ldots$$

The sequence does not have to start with 1 either, it can start with any number. For example, the following geometric series starts with 5 and has a growing factor of 3:

5, 15, 45, 135, 405, …

However, note that if we do start with 1, the nth term of the sequence is just the $n-1$ power of the growing factor. For example, in 1, 2, 4, 8, 16, 32, 64, 128, … the sixth term is $2^5 = 32$.

Geometric Series

A series is the sum of all sequence's terms. For example, the series representation of the first example above (the geometric sequence with factor of 2) would be:

1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + …

As you can see from the previous example, series can get very big very fast. This example is a divergent series, meaning it is effectively impossible to add all the terms. We might also say that their sum is infinity.

Since we are attempting to add an infinite number of terms, it might seem like any sequence would add to infinity, or diverge. However, in some cases the sum can in fact be calculated, the result is an actual number. For a geometric series, this happens when the growing factor is less than 1. More formally:

The geometric series $1, r, r^2, r^3, \ldots$ *converges if and only if* $|r| < 1$ *and in that case*

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Where the symbol $\sum$ is used as short notation for a sum of many (in this case infinite) numbers.

So, taking our previous example with $r = \frac{1}{2}$, we have

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$$

If we subtract 1 from both sides, we see that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots = 1$$

Objectives

- To discover the hidden lives of Black mathematicians from Ohio State.
- To realize that the valuable work and stories of some people remain hidden and we have the power to rectify this.
- To introduce series and sequences.
- To recognize that the sum of a sequence can be a small number even if there is an infinite number of terms being added.

Links with Standards

SOCIAL SCIENCES	
Grade	Standard
2	4. Biographies can show how peoples' actions have shaped the world in which we live.
4	8. Many technological innovations that originated in Ohio benefitted the United States
MATHEMATICS	
Grade	Standard
3.G.2	Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.
4.OA.5	Generate a number or shape pattern that follows a given rule.
Practice 1	Make sense of problems and persevere in solving them.
Practice 2	Reason abstractly and quantitatively.
Practice 3	Construct viable arguments and critique the reasoning of others.
Practice 8	Look for and express regularity in repeated reasoning.

Materials

Each student needs:

- 2 Graham crackers
- A napkin or plate to put the crackers
- Worksheet
- Pencil or writing utensil

Note: The activity uses graham crackers for added fun. However, if this does not work well for your class, crackers can be substituted by rectangles of cardboard, cardstock or even scrap paper. The rectangles don't need to be a specific size either, as long as they are not too small, and two of them side-by-side fit within a sheet of letter paper. You might also need scissors in this scenario, although tearing with hands is fine too.

The worksheet simply consists of a square that matches the size of two graham crackers side-by-side. Before the activity, verify that the crackers you bought do match the worksheet, since there could be some variation based on brand. If necessary, create your own version of the worksheet with a square that fits the size of your crackers. Another alternative is to have students start by tracing a square around the border of their two side-by-side graham crackers, either on a loose sheet of paper or on their notebook.

Opening/Motivation (25 minutes)

1. Introduce Dr. Rada Higgins to students using materials gathered from the Hidden Figures Revealed project.
2. Tell the class that in her dissertation, Dr. Higgins studied sequences and what happens if you try to add all the terms in a sequence. Ask the students to recall what a sequence is and to propose some examples. Write the examples on the board. If examples of geometric sequences don't come up, make sure to suggest one yourself.
3. Next, ask the class what would happen if we added all the terms in the sequences they gave as examples. At first, they might not think much of the question. They are likely to simply add the terms on the board. Point out that, while that is a good start, we are not done, since there are more terms in the sequence, it is just that we did not write them all.
4. When they give you the sum of the terms of a given sequence on the board, write the next term and ask them to update the sequence. Do this a few times to highlight the idea that we would have to continue adding.
5. After this discussion, the goal is that the class concludes that it doesn't make sense to add an infinite number of terms, that you would never finish. At this point, reveal that, despite what our intuition might tell us, sometimes it is actually possible to add ALL the terms on some sequences. These cases are what Higgins studied and we are going to explore one of the examples she, and many mathematicians, have worked on.

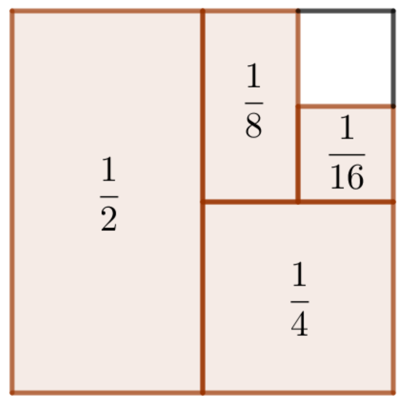
Procedures (15 minutes)

1. Distribute crackers on napkins/plates and worksheets.
2. Tell students to take one full graham cracker and place it inside their square, flushed left. They should also draw a line to complete the border of the cracker.
3. Ask the class what fraction of the square covers the cracker. They should identify that it is ½. Have them to write that fraction on the paper right underneath the cracker, lifting it for a moment and then placing it back on the square.
4. Next, have students split their other graham cracker in half and place it in the open half of their square, flushed down. What fraction of the square does this piece of cracker correspond to? Again, they should trace a line to mark the border of this graham cracker and write the fraction right underneath it.

Note: If you are using paper or cardstock instead of crackers, the students should fold the piece in half and then tear or cut.

5. Have students split what's left of their graham cracker in half, trace its border, and write the fraction.

6. At this point, the remaining cracker should be the "unit" graham cracker. While it will be more challenging, get them to try to split it one last time and repeat the process of tracing and writing the fraction. They should have a picture like the following on their worksheets, with the brown rectangles covered with crackers:



Note: If you are using paper instead of crackers, you can continue the halving process as much as the material permits.

Discussion (20 minutes)

Now, the class will reflect upon the process.

1. First, notice that if they put the remaining piece of cracker on the square, it would fill up the entire square. If they could continue splitting their graham cracker pieces in half over and over, and adding the pieces to the diagram, would it ever take up more space than inside the square? Why or why not? Since they would only continue splitting that same piece, but never add more cracker, then it cannot overfill the square, no matter how many more times you split it.

2. Have students remove the graham cracker pieces from their worksheet. Next, they are going to put the crackers back on the square but at the same time, they will write the corresponding sum of fractions. So they would place the biggest piece of cracker and write ½, since that is the area of the square currently covered. Then place the second biggest piece and write "+¼" and so on. After putting back all the pieces (still leaving one of the smallest pieces out) they should have:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$$

3. Ask them to imagine that they cut the cracker in half one more time and, with their pencil, fill the area that this piece would take up the square. What fraction of the square would this piece be? Add it to the sum:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$$

4. Do this yet one more time to get

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$$

5. Then write on the board

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\cdots$$

   and challenge the class to hypothesize what the sum of ALL the terms of the sequence adds up to. Remind them that these are fractions of the square. If you continue adding further smaller fractions to the square and the sum, what will you get? Hopefully most students will see that the square will be more and more full, but never overflow.

6. Conclude by saying that this sequence adds up to 1.

Optional: If your students have access to devices such as tablets, laptops, or computers, have them explore the interactive application on the following link. They can move the n slider to add more and more pieces to the square and notice how you never overflow it.

https://demonstrations.wolfram.com/AConvergingGeometricSeries/

7. Let them eat their crackers, answer any questions they might have, and encourage them to share how they felt with this activity and what they think it might have been like for Dr. Higgins as she was exploring this topic.

References

- Higgins, Rada Ruth. *On the Asymptotic Behavior of Certain Sequences*. Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Graduate School of The Ohio State University. 1974
- Higgins, Rada. Resources on *Teachers Pay Teachers*. https://www.teacherspayteachers.com/Store/Rada-Higgins
- Wikipedia. *Geometric Series*. https://en.wikipedia.org/wiki/Geometric_series