Picture Hanging Puzzle

Grade or age level: Grades 7-12 **Time:** 100 minutes **Form of work:** Small groups

Background

Guy Hogan is one of the seven Black mathematicians that have obtained a PhD in mathematics at The Ohio State University. His dissertation "Variations on the H_p Problem for Finite p-Groups" explored finite group theory, especially groups with a prime power number of elements (i.e. $3^2 = 9$ elements). See below under "Group Theory" for more details about groups. In this activity, students explore group theory through collaborative puzzles.

Puzzle Introduction

Imagine you are hanging a picture on three nails. The picture hanging puzzle asks: how can you hang the picture so that removing any of the three nails will cause the picture to fall? For this puzzle, we assume that there's a piece of string tied to the two upper corners of the picture. The natural way to hang this would be to simply place the string resting over the three nails. However, with this method, removing any one of the nails will not cause the picture to fall, as the string will still be supported by the other two nails. Thus, the goal is to find a more complicated way of wrapping the string around the three nails so that, when either of the nails is removed, the string unwraps itself from the other nails, and the picture falls. A solution to this puzzle can be found using a group theory model.

Group Theory

In mathematics, a group is a set of elements and an operation defined in the set that meets certain conditions:

- When two elements of the group are operated, the result is also an element of the group. This might seem like an obvious property, but there are familiar examples that do not meet this condition. For example, think of the set of the whole numbers and the division. Some of the elements on this set, when dividing one by the other, give a fraction result, a number that is not on the set of the whole numbers. Hence, division in the whole numbers does not make a group.

- The operation is associative, which means when operating three elements in a given order, x, y, z, there's no difference if you first operate x and y and then the result with z, or operate first y with z and then the result with x. This is expressed as $(x \circ y) \circ z = x \circ (y \circ z)$, where $\circ$ represents the group operation. Note that the order of the elements is important. We are only changing which pair of the operation we solve first, not the order of the elements themselves. Operating x and z would not be valid, as y is in the middle of them. Associativity is important because operations are defined for two elements only, so three or more elements cannot be operated at once.
- There is an identity element, which we can call I, such that when operating it with any element, the result is that same element: $I \circ x = x \circ I = x$. Note that it doesn't matter if the identity goes before or after the element.
- Every element in the group has an inverse, such that when the element and its inverse are operated, the result is the identity.

Examples:

- The set of the integers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, is a group under the addition operation.
  - Any time you add two numbers, the result is another integer number.
  - The addition is an associative operation, as we know.
  - 0 is the identity element, because adding 0 to any integer is the same integer.
  - Every integer has an inverse, which is the negative of itself. For example, the inverse of 2 is –2 since 2 + (-2) = 0; the inverse of 0 is itself since 0 + 0 = 0.
- As a non-example, the integers are not a group under the multiplication operation, since there are no inverses within the same set. 1 is the identity element, since 1 times any integer is the same integer. However, the inverse of 2 would be ½, since $2 \times \frac{1}{2} = 1$, but ½ is not an element in the set, since it is not an integer.

Group Theory is an extensive field of mathematics that studies these mathematical structures called groups.

It is important to note that, in general, the operation on a group does not have to be commutative. The example we cited before, $(\mathbb{Z}, +)$, is in fact commutative, but, as we will see later, the group that models the hanging puzzle is not.

Commutator

In the context of non-commutative groups, there is one very important object called the commutator. The commutator of two group elements x and y is the element $x^{-1}y^{-1}xy$, where we are using the -1 exponent to represent inverses (i.e. x^{-1} is the inverse of x and y^{-1} is the inverse of y).

Note that, in a commutative group, this operation simply gives the identity:

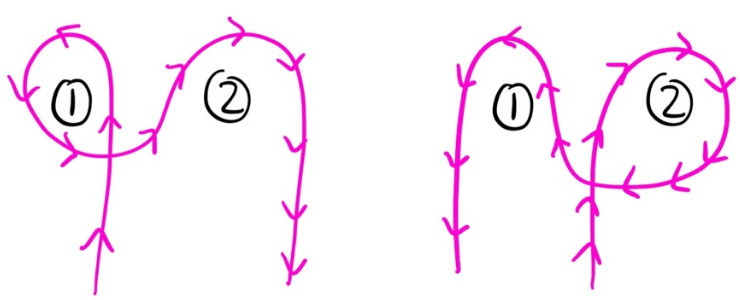
$$x^{-1}y^{-1}xy = x^{-1}xy^{-1}y = (x^{-1}x)(y^{-1}y) = (I)(I) = I$$

Puzzle as a Group

Hanging a picture around nails forms a group where the elements are the actions of the string: counterclockwise or clockwise over a specific nail; for example, clockwise over the first nail. The operation is composition (doing one action after another). The actions under composition satisfy the three conditions of a group:

1. Associativity: regardless of parentheses, the order of the actions will not change, so we will end up with the same result.
2. Existence of identity element: doing nothing is the identity because doing nothing then doing any action will be the same as only doing the action.
3. Every element has an inverse element: given any action, we can undo the action by wrapping the string in the opposite direction on the same nail. The action that undoes our initial action is the inverse element of our initial action. For example, the inverse action of going clockwise around a nail is going counterclockwise around the same nail.

Although these actions are associative, note that they are not commutative. For example, consider a puzzle with two nails: nail 1 and nail 2. Suppose we first wrap the string counterclockwise around nail 1 then clockwise around nail 2 (depicted left in the figure below). This sequence of actions does not yield the same result as executing the actions in the opposite order: first wrapping the string clockwise around nail 2 then counterclockwise around nail 1 (depicted right in the figure below).



The first sequence of moves (left) wraps the string completely around the first nail, but the second sequence of moves (right) wraps the string completely around the second nail.

Puzzle Solution

First, we will consider the puzzle with two nails and the condition that removing either nail causes the picture to fall. The solution to this puzzle is to weave an action on one nail between two inverse actions on the other nail as so:

$$x^{-1}y^{-1}xy$$

where x denotes wrapping the string clockwise around the first nail (so x^{-1} denotes wrapping the string counterclockwise around the first nail), and y denotes wrapping the string clockwise around the second nail (so y^{-1} denotes wrapping the string counterclockwise around the second nail). Notice that this is precisely the commutator of x and y. When we weave an action on the other nail between two inverse actions, this prevents the two inverse actions from cancelling out each other. Hence, this sequence of actions will actually hang the picture (actions do not all cancel out). Moreover, when we remove a nail (remove the action between the two inverse actions), the two inverse actions on the remaining nail will cancel out. Thus, the picture will fall.

To solve the original puzzle with three nails and the condition that removing any nail causes the picture to fall, we build upon the previous solution to the puzzle on two nails. We take the commutator of the previous solution and the clockwise action on the third nail (denoted by z):

$$y^{-1}x^{-1}yxz^{-1}x^{-1}y^{-1}xyz$$

Note that $y^{-1}x^{-1}yx$ is the inverse of the previous solution, so if we remove the z and z^{-1} (by removing the third nail), the remaining actions will cancel out, and we are left with nothing. Thus, removing the third nail will cause the picture to fall. If we remove either the first or second nail, as seen previously, those removals will each cause the previous solution to cancel out, so we are left with $z^{-1}z$, which cancel out to be nothing. Therefore, removing any of the nails will cause the picture to fall.

This pattern can be extended to solve the puzzle with n nails and the condition that removing any nail causes the picture to fall. A solution is to take the commutator of the solution of the same puzzle with $n - 1$ nails and an action on the new nail.

Objectives

- To discover the hidden lives of Black mathematicians from Ohio State
- To realize that the valuable work and stories of some people remain hidden and we have the power to rectify this.
- To explore the idea of inverses.

- To explore some mathematical ideas that appear in puzzles.

Links with Standards

SOCIAL SCIENCES	
Grade	Standard
2	4. Biographies can show how peoples' actions have shaped the world in which we live.
4	8. Many technological innovations that originated in Ohio benefitted the United States
MATHEMATICS	
Code	Standard
6.EE.3	Apply the properties of operations to generate equivalent expressions.
6.EE.4	Identify when two expressions are equivalent.
7.NS.1	Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.
Practice 1	Make sense of problems and persevere in solving them.
Practice 2	Reason abstractly and quantitatively.
Practice 3	Construct viable arguments and critique the reasoning of others.
Practice 8	Look for and express regularity in repeated reasoning.

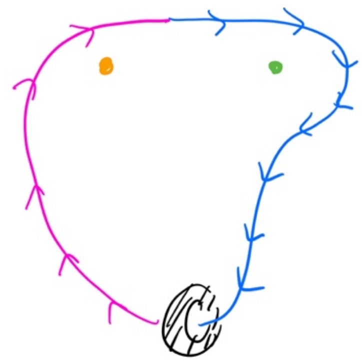
Materials

- Guy Hogan poster
- One game set per group of 3-4 students. Each game set includes:
  - 1 pinboard placed vertically
  - 1 washer on a loop of yarn
  - 3 different color thumbtacks (we will reference orange, green, and blue in this document, but the educator can use other colors as available)
  - 3 sets of 5 addition (+) cards, each set is a color corresponding to a thumbtack
  - 3 sets of 5 subtraction (–) cards, each set is a color corresponding to a thumbtack
- A rope (for example a jumping rope)

Note: We used magnetic thumbtacks on a magnetic whiteboard. The educator can adapt the materials to what is available in the classroom.

Opening and Motivation (25 minutes)

1. Introduce Dr. Guy Hogan to students using materials gathered from the Hidden Figures Revealed project.
2. Introduce the activity by mentioning that it is related to group theory, a field that Dr. Hogan studied. Students will likely be curious about what this group theory field is about. We suggest leaving that discussion for the end, once they had a chance of exploring the example through the activity.
3. Explain that the activity is about hanging a weight on thumbtacks. Demonstrate a "normal" way of hanging a weight on two thumbtacks (orange and green) looping the string over both thumbtacks clockwise, as depicted in the image. Show that removing any thumbtack does not cause the weight to fall. Explain that the puzzle is about hanging the weight so that removing any of the thumbtacks causes the weight to fall.



Note: To demonstrate in front of the class, we suggest having one or two student volunteers. Their extended arms will play the part of a thumbtack and you should use a rope instead of a thin string. This would allow the whole class to see what is being demonstrated.

4. Ask for suggestions on how to solve this puzzle. Conclude that this puzzle is complicated. Then, explain that mathematicians created a model for this puzzle to make it easier to understand and solve.

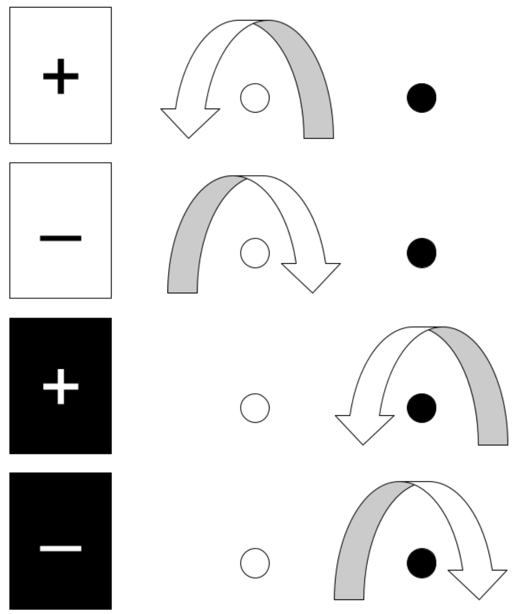
Note: Depending on the characteristics of your group, you can allow a short time (~5 min) for students to experiment in groups. Consider that sometimes students get caught in this and it is hard to get their attention back. Allowing a short time for experimentation, with the promise that they would go back to it trying after learning some useful things, worked well for us.

Procedures (75 minutes)

Exploration (30 minutes)

1. Organize students into groups of 3 or 4 students.
2. Explain that they will explore the mathematical model with cards. The cards represent different actions of hanging the string. The color of the card symbolizes which color thumbtack to wrap the string around. The sign of the card (+ or –) symbolizes whether to

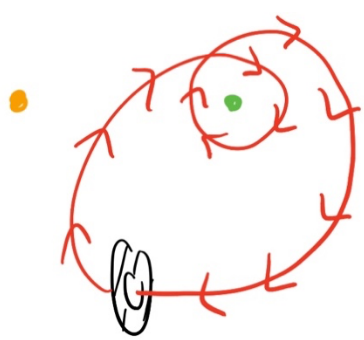
wrap the string clockwise (+) or counterclockwise (–) around the thumbtack. For example, if there were a black and a white thumbtack, then we have the following actions:



3. Explain that the idea is to use the cards to give instructions on how to hang the string. A sequence of cards represents a sequence of moves with the string around the tacks. Demonstrate a few sequences so that students get familiar with the model. For example:
   - The "normal" way of hanging the weight, shown on the previous page, is

     + +

   - The sequence + + means wrapping the string clockwise around the green thumbtack twice as shown in the diagram.



   - The sequence + – means wrapping the string clockwise around the orange thumbtack and then counterclockwise around the same tack, thus causing the weight to fall. We can say that cards of the same color with opposing symbols cancel out. However, this only happens if the two cards are right next to each other.

- For example, the sequence + + – does not cause the weight to fall.
- Show the sequence + + – too and point out that, since the two last cards cancel out, this sequence is equivalent to the single move +.
- What sequence would we need to perform to cancel out the whole sequence + + –? Undoing a single move is easy, but after you've done a series of moves, it gets harder to undo. The cards actually help with the undoing process, because they help us remember what we did. Help students notice that to successfully undo a series of moves we need to undo them starting from the last one. Hence, if we do + + –, then we know that the last move we did was a counterclockwise around the orange thumbtack. To undo it, we need to go clockwise around the same tack. Next we undo the green move and last the very first move. In conclusion, to cancel the sequence + + – we do + – –:

(+ + –)(+ – –) -> the weight falls!

4. Do more examples if you feel it is necessary. Once students understand how the card model works, propose the following sequences:

   + + – –   + + – –   + + – –

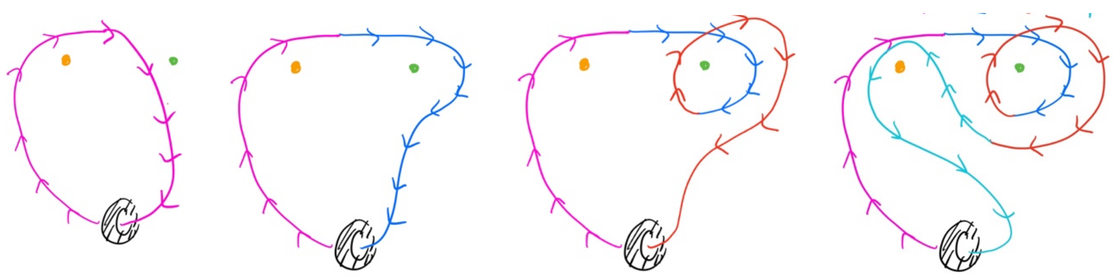
   Ask students if they can predict whether any of these will result in the weight falling and explain their reasoning.
5. Distribute the materials (just two thumbtacks and two colors of cards for now) and have students test their hypothesis regarding the previous sequences. As they are doing that, walk around the room and verify that they are using the model properly. They should conclude that in the first case, the weight falls because the two interior cards cancel out, making it be as if only the first and the last card were used, but these two cancel out as well. Because of that, the sequences + + – – and + – are both equivalent to not doing anything (no cards), and also equivalent with each other. This does not happen in the other two sequences because adjacent cards with opposite signs are different colors, so no cards cancel out.

Variation: You can skip steps 6 and 7 below to shorten the activity a bit.

6. To gain more practice and familiarity with the moves, now each group shuffles their cards, draws 10, and performs the sequence of actions indicated by them. Have them place the cards face-up on the table in the order that they were drawn, so they have a clear reading of the sequence.

7. When they are done, have the different groups share what they noticed and discuss the following questions as a class.
   a. Did the weight fall at any point in the process?
   b. After the 10 actions/cards, was the weight hanging from one or more of the pins or was it completely untied from them?
   c. For those cases when the weight fell, why do they think that is?
   d. Did any of the cards canceled out? Is there a shorter sequence of actions that would hang the weight the same way as the 10-card sequence? If so, what is it?
   e. What else did they notice?

   Emphasize that the way the string and the weight are tied up on the pins is the result of the 10-card sequence.
8. As a last step in the exploration stage, we will introduce removing thumbtacks. Start by asking them to hang the weight according to the sequence + +. Instruct them to remove the orange tack and guide them to notice that the result is equivalent to removing the orange card (keeping the green only).
9. Put the orange thumbtack back on and again arrange the sequence + +. Now have students remove the green tack and notice that the result is equivalent to removing the green card.
10. To further explore this property have them try the more complicated sequence + + + –, remove the orange tack, and verify that the result is equivalent to removing all orange cards.



11. Reset the tack and the sequence and this time remove the green tack. Students should again notice that the result is equivalent to removing the green cards and that, in this case, the weight falls because the resulting sequence is + –, so the remaining orange cards cancel out.

The Puzzles (30 minutes)

1. With all that they have learned and noticed so far, challenge the groups to come up with a sequence of cards that effectively hangs the weight, but that after removing any one of the

thumbtacks the weight falls. You can give them the hint that the card sequence has to be such that if you remove all the cards of one color, the remaining cancel out.

2. If none of the groups finds a solution, reveal that the simplest one is + + – –. Make sure they understand why this solution works. They should notice that the trick is to take two cards that cancel out and stick one of the other color in the middle.

Note: Another solution is to repeat the previous one twice: + + – – + + – –. Then, if we remove the green tack, we are left with + – + – which cancels out to nothing. Similarly, if we remove the orange tack, we are left with + – + – which also cancels out to nothing.

3. Next, distribute the other thumbtacks and the rest of the cards so students can work on the three nail puzzle. Again, the goal is to come up with a sequence of cards that result in hanging the weight but that removing any of the three thumbtacks causes it to fall.
4. Encourage the groups to try to build their solution from the 2-nails solution, thinking carefully about canceling out cards.
5. If a group finishes early, ask them to record their solution on a sheet of paper or by taking a picture. Then, they can try another puzzle from the additional puzzles annex.
6. Most likely, not all groups are going to be able to find a solution. Give some reasonable time (10 minutes max) and the build the solution as a group. We suggest building the solution in the following way:

   With two tacks, we got our solution by taking two cards that we knew cancelled out, + and –, and putting a green card in the middle. Then we put the opposite green card at the end and this did the trick. Well now we have a sequence of four cards that work for removing any one of two tacks. Following the same idea as before, we need to find a sequence that cancels this one. This is easy to do, we just need to do the opposite of each card backwards:

+ + – –	First we need to cancel the last move, so we do +.
+ + – – + equiv. + + –	We basically undid the last move, so now we need to "undo" the last one left, so we do +.
+ + – + equiv. + +	To cancel the remaining two moves we need to do – first and then –.

   Hence, the inverse of + + – – is + + – –.

   If we do these two sequences one after the other, everything will cancel out and the weight will fall. We incorporate the third thumbtack by putting a move on that tack in between these two sequences, thus blocking them from cancelling out. Next, just as we did before we plug the opposite move on the third tack at the end of the whole sequence:

+ + − − + + + − − −

With the class, check that when removing the cards of any color, the rest cancel out. Then perform the sequence on string and verify that it works as solution to the puzzle.

Closure (15 minutes)

1. Encourage students to share their thoughts and feelings about the activity. Did they feel the card model was actually useful to solve the puzzle or at least to understand why a solution make sense?
2. Ask students if they thought what they were doing was math. Explain that this puzzle relates to a part of mathematics called group theory. Emphasize that doing math isn't just solving complicated equations on paper, but it can be playing around with puzzles.
3. The key idea in the activity is inverses. Bring this point up if the students don't mention it themselves. You can connect to other inverses you have studied in the classroom, such as the reciprocal of a fraction or how they use inverses when solving an equation.
4. From here, you can briefly talk about groups. Explain how they are a widely used structure in mathematics that has three important characteristics: associativity, an identity, and inverses. Mention how they have in fact been using some groups in their math class since elementary school (the whole numbers and their addition, the rational numbers and multiplication). However, groups don't have to involve numbers. Many puzzles, like the one we explored here, and the Rubik's cube make up groups. Instead of numbers, in our "puzzle group" the elements are cards, more precisely, the operation of wrapping the string around the thumbtack in clockwise or counterclockwise direction. Relate back to Dr. Hogan by mentioning that he studied other groups.

Variation: The talk about groups can go as deep or be as superficial as you and your class feel comfortable. However, we do suggest you mention the word and point out the similarity between the cards and the addition operation, as well as highlight how they are different because the cards are not commutative, while the usual addition is. Some other points you can make about groups in relation with the activity are:

- The trick we used in this activity of putting something in between two inverse moves is very useful in group theory and it is also used in solving the Rubik's cube. It is called a commutator.
- Turning a coin makes a group too. The two "things" in the group would be the actions of turning over the coin and doing nothing. The "operation" would be doing the actions in

order. Doing nothing is the identity element because doing nothing then turning over the coin is the same as just turning over the coin. The inverse to turning over a coin is itself (turning over the coin twice is the same as doing nothing).

- In the Rubik's cube, the elements are the moves on the cube. The operation again is doing one move after another in certain order.

Reference

Demaine, E., Demaine, M., Minsky, Y., Mitchell, J., Rivest, R. and Pătraşcu, M., 2013. Picture-Hanging Puzzles. Theory of Computing Systems, 54(4), pp.531-550.

Additional Puzzles

(from Demaine, 2013)

1. **(2-out-of-3)** Hang a picture on three nails so that removing any two nails fells the picture, but removing any one nail leaves the picture hanging.
2. **(1+2-out-of-3)** Hang a picture on three nails arranged along a horizontal line so that removing the leftmost nail fells the picture, as does removing the rightmost two nails, but removing one of the two rightmost nails leaves the picture hanging.
3. **(1-out-of-4)** Hang a picture on four nails so that removing any one nail fells the picture.
4. **(2-out-of-4)** Hang a picture on four nails so that removing any two nails fells the picture, but removing any one nail leaves the picture hanging.
5. **(3-out-of-4)** Hang a picture on four nails so that removing any three nails fells the picture, but removing just one or two nails leaves the picture hanging.
6. **(2+2-out-of-2+2)** Hang a picture on two red nails and two blue nails so that removing both red nails fells the picture, as does removing both blue nails, but removing one nail of each color leaves the picture hanging.
7. **(1+2-out-of-2+2)** Hang a picture on two red nails and two blue nails so that removing any one red nail fells the picture, as does removing both blue nails, but removing just one blue nail leaves the picture hanging.
8. **(1+3-out-of-3+3)** Hang a picture on three red nails and three blue nails so that removing any one red nail fells the picture, as does removing all three blue nails, but removing just one or two blue nails leaves the picture hanging.
9. **(1+2-out-of-3+3)** Hang a picture on three red nails and three blue nails so that removing any one red nail fells the picture, as does removing any two of the blue nails, but removing just one blue nail leaves the picture hanging.
10. **(1+1-out-of-2+2+2)** Hang a picture on two red nails, two green nails, and two blue nails so that removing two nails of different colors (one red and one green, or one red and one blue, or one green and one blue) fells the picture, but removing two nails of the same color leaves the picture hanging.

+	+	+	+	+
+	+	+	+	+
−	−	−	−	−
−	−	−	−	−