The Ohio State University

BLACK MATH HISTORY
HIDDEN FIGURES REVEALED

Sum of Squares Game

Grade or age level: Grades 3 – 7 **Time:** 90 minutes **Form of work:** Small groups

Background

Thyrsa Svager is one of seven Black mathematicians that have obtained a PhD in mathematics at The Ohio State University. On her master's thesis, she explored the problem of describing the numbers that can be represented as the sum of two, three or four squares. Here, the word "squares" refers to square numbers or perfect squares, that is numbers that are the resulting product of an integer with itself.

In this activity, students explore some of the concepts on Dr. Svager's thesis through a two-player board game. An alternative, simpler game is offered for younger students.

Sum of two squares

To check if an integer can be represented as the sum of two squares, one has to look at its prime factorization. It should not contain any primes that are 1 less than a multiple of 4, unless said prime appears an even number of times. Formally, an integer greater than one can be written as a sum of two squares if and only if its prime decomposition contains no factor p^k, where $p \equiv 3$ (mod 4) and k is odd.

Examples:

- $6 = 2 \times 3$ cannot be written as the sum of two squares because its prime factorization contains a single 3 (and 3 is 4 – 1).
- $18 = 2 \times 3 \times 3$ can be represented as sums of two squares, because even though 3 is 1 less than a multiple of 4, it appears two times.
- $130 = 2 \times 5 \times 13$ can also be represented as sums of two squares because it does not contain any primes that are 1 less than a multiple of 4 (both 5 and 13 are 1 more than a multiple of 4, 5 = 4 + 1 and 13 = 12 + 1).

Sum of three and four squares

The only case where a number cannot be written as the sum of three squares is when it's the product of a power of 4 times a number that is one less than a multiple of 8. Formally, the Legendre's three-square theorem states that the only numbers that cannot be written as sum of three squares are those of the form $4^a(8b + 7)$ with $a, b \geq 0$.

Examples:

- 4 and 16 can be written as the sum of three squares because, even though they are both powers of 4, they are not multiplied by a number that is one less than a multiple of 8.
- 7 cannot be written as the sum of three squares because 7 is one less than a multiple of 8 and 1 is a power of 4 ($7 = 1 \times 7 = 4^1(8(0) + 7)$).
- $16 \times 7 = 112$ and $4 \times 15 = 60$ are other examples of numbers than cannot be represented as sums of three squares.

Finally, Lagrange's four-square theorem states that all numbers can be represented as the sum of four squares.

Some example representations

Number	Two Squares	Three Squares	Four squares
4	0^2+2^2	$0^2+0^2+2^2$	$1^2+1^2+1^2+1^2$
6		$1^2+1^2+2^2$	$0^2+1^2+1^2+2^2$
7			$1^2+1^2+1^2+2^2$
16	0^2+4^2	$0^2+0^2+4^2$	$2^2+2^2+2^2+2^2$
18	3^2+3^2	$1^2+1^2+4^2$	$0^2+1^2+1^2+4^2$
21		$1^2+2^2+4^2$	$0^2+1^2+2^2+4^2$
25	3^2+4^2	$0^2+0^2+5^2$	$2^2+1^2+2^2+4^2$
36		$2^2+4^2+4^2$	$1^2+1^2+3^2+5^2$
50	5^2+5^2	$3^2+4^2+5^2$	$1^2+2^2+3^2+6^2$

If you need to quickly find a two, three, or four squares representation, we recommend using the following website, which is a powerful calculator: https://www.wolframalpha.com/ On the search bar, type the equation you need to solve, namely $a^2+b^2+c^2+d^2=$ the number you are searching for, using as many squares as you need. Press enter and wait for results to load. It will give more information that you need but look for the "Integer solutions" subtitle to find the representation you need.

Lesson Details

Objectives

- To discover the hidden lives of Black mathematicians from Ohio State.
- To realize that the valuable work and stories of some people remain hidden and we can do something to change that.
- To practice multiplying and adding.
- To introduce the concept of a square number.
- To identify patterns.

Links with Standards

SOCIAL SCIENCES	
Grade	Standard
2	4. Biographies can show how peoples' actions have shaped the world in which we live.
4	8. Many technological innovations that originated in Ohio benefitted the United States

MATHEMATICS	
Code	Standard
3.MD.7	Relate area to the operations of multiplication and addition.
4.OA.4	Find all factor pairs for a whole number in the range 1-100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1-100 is prime or composite.
4.OA.5	Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.
Practice 1	Make sense of problems and persevere in solving them.
Practice 2	Reason abstractly and quantitatively.
Practice 3	Construct viable arguments and critique the reasoning of others.
Practice 8	Look for and express regularity in repeated reasoning.

Materials

- Thyrsa Svager poster
- Thyrsa Svager video
- Projector
- Paper and pencil
- One game set per group of 3 or 4 students. Each game set includes:
    - Number cards (1 to 50)
    - Square cards (19 total in seven different sizes)

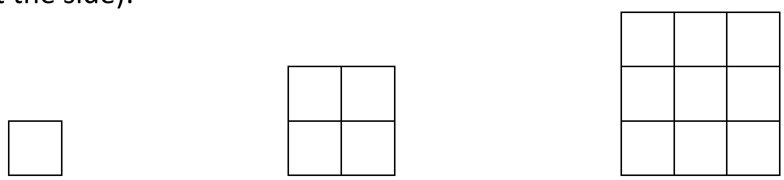
Opening Activities/Motivation (30 minutes)

1. Introduce Dr. Thyrsa Frazier Svager using the slides that you can access on this link. The set of slides contain the following short biography of her and a series of question to reflect upon her life and circumstances. Discuss the questions with the class.

    Born in 1930 in Wilberforce, Ohio, Thyrsa Frazier Svager achieved what few African-American women of her generation have in the field of education. She was one of the first African-American women in the United States to earn a Ph.D. in mathematics. Her mother (a professor) and her father (a statistician) instilled the importance of education in her from a young age. Thyrsa Svager graduated from the Wilberforce University Preparatory Academy at the age of 15, going on to complete her undergraduate studies at Antioch College in Yellow Springs, and earning both her master's degree and doctorate in mathematics at The Ohio State University. Dr. Svager worked as a statistical analyst at Wright-Patterson Air Force Base and as an instructor at Texas Southern University in Houston. She spent most of her career at Central State University, where she not only served as professor, but also as administrator, Dean, Provost, and Interim President.

- Dr. Svager was 1 of 4 black students at her college at the time of enrollment. How do you imagine she felt? What would you have done if you had been her classmate?
- Do you think her parents' professions might have influenced her career choice?
- What are your parents' occupations? Would you like to work on the same fields?

2. Introduce the activity by mentioning that the game they are going to play is inspired by Svager's master thesis work "The representation of integers as sums of two, three or four squares." Further, explain what the word "squares" means in that context: a square is a number that results from the area of a square whose sides have an integer length. We suggest using the following examples, emphasizing that the square numbers are 1, 4, and 9 (the area, not the side):



Side: 1 | Area: $1 \times 1 = 1$

Side: 2 | Area: $2 \times 2 = 4$

Side: 3 | Area: $3 \times 3 = 9$

   To gauge understanding, ask students to mention another square number. Then ask if 5 (or any other number in between 1, 4, and 9) is a square number. Invite them to share their thought processes in concluding that 2, 3, 5, 6, 7, and 8 are not square numbers.

Variation: Depending on the characteristics of the group, you can have students notice that areas of squares are always calculated as a repeated multiplication (the number times itself). Thus, another way of defining square numbers is that they are numbers that result from the multiplication of a whole number with itself.

3. Continue explaining the title of the thesis, now focusing on the part about the sum. Svager was studying three related problems: numbers that result from adding two squares, numbers that result from adding three squares, and numbers that result from adding four squares. For example, if we sum 1 + 4 (the areas of a square with side 1 and a square with side 2, respectively) the result is 5. Ask them to come up with another example.
   Point out that not all numbers can be written as the sum of two squares; Svager was studying which ones can. Give an example of this, like 3 which can only be broken up into the sum of two positive integers as 3 = 1 + 2, but 2 is not a square number, so it doesn't work. However, 3 can in fact be written as the sum of three squares, because 3 = 1 + 1 + 1 and 1 is indeed a square number. Challenge them to give further examples of sums of three and four squares.

Note: Since $0 \times 0 = 0$, 0 is in fact a square number. Thus, for example, 5 can be written as the sum of three squares as 5 = 0 + 1 + 4. However, 0 is not the area of a square, which can be confusing to younger students. We suggest you don't bring it up, and only explain this point if the students themselves bring it up.
The game leaves out the cases that use a 0 in their sum of squares representation. In the case of two squares, those are simply the square numbers themselves.

4. Finally, mention that master theses are often an exhaustive review of previous work done by others in a specific field. In the case of Svager, she chose to survey this topic, but the problems she analyzes were solved before the 19th century.

Procedures (25 minutes)

Warming Up (10 minutes)

1. Before playing the game, spend a little more time getting familiar with the problem Svager studied. To do so, organize the class into groups of 3 or 4 students.
2. Give each group a set of square cards.
3. As a group, ask them to combine the cards in pairs in as many ways as possible and to write down the sum of each pair.
4. Make sure that they understand that, after registering a pair, they can undo that pair and match these cards with other cards.
5. Once the groups have a good amount of pairs written down, they can move on to repeat the activity now with trios of cards.

Game (15 minutes)

1. Students can play in the same groups they were in.
2. Ask them to put away the list they made on the previous step and then distribute the number cards, which simply have the numbers 1 to 50 on them.
3. To prepare for the game, the groups should mix up the number cards and put the pile facing down. They should also spread all their square cards on the table, face up.
4. Players take turns to draw a number card from the deck, which will be placed on the table facing up, where all players can see it.
5. At this point, all players will quickly try to find two square cards that add up to the number card. The first player to grab and present two such cards, claims the number card.
6. Players cannot pick up cards and keep them on their hands while still looking for a card. Cards can only be picked up when the player has located the two that add up to the number card.
7. The person who claims the number card keeps that card, but the square cards must be returned to the table.
8. If the number card cannot be expressed as the sum of two squares, the group needs to agree that there is no solution. In that case, the number card is discarded.

Variation of previous step: A player can claim the number card if they correctly explain why that number cannot be written as the sum of two squares. The other players as a group will judge whether the explanation is satisfactory. The group can ask questions from the player explaining, to clarify the argument. In this case, the teacher should motivate students to judge fairly.

9. The game is over when there are no more number cards on the pile. The winner is the player with the most claimed cards.

Let every group finish at least one full round. Depending on available time, they can either stop there, play more rounds, or play alternative versions of the game with sums of three or four squares or allowing for all possibilities on the same game (on a round, players can present either two, three, or four squares to add up to the number card). When they are done playing, have them keep the discarded cards separate for further analysis during the next stage. If they are going to play the other versions of the game, ask them to write down the numbers on the cards they discarded on each game.

Closure (35 minutes)

1. Ask the students to say the numbers on the cards they discarded and write them on the board. These should be the numbers which cannot be written as sums of two squares. For each number they give, make sure that the whole class agrees that the card was correctly discarded. If someone disagrees, they should give a way of writing that number as the sum of two squares. Promote an environment where students feel comfortable sharing their thoughts even if they are incorrect, and where they can politely correct each other. You might want to point out that it is easy to make these mistakes as we are playing, especially since the game is fast paced; finding the correct sum of squares is a difficult task. Explain that is the reason why you are dedicating some time to check.
2. Correct any mistakes the class might have made. The numbers on the board should be 1, 3, 4, 6, 7, 9, 11, 12, 14, 15, 16, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, 35, 36, 38, 39, 42, 43, 44, 46, 47, 48, and 49.
3. Ask the class to look for any patterns on these numbers. One thing they are likely to spot is that almost all multiples of 3 and all multiples of 7 cannot be written as the sum of two squares. The only exceptions are 18 and 45, which are multiples of 9. If no one in the class identifies these patterns, you can reveal it to them.

Note: 9 and 49 can be written as the sum of two squares, but the representations would need 0. The list also includes all multiples of 11 and 19, but that is likely to go unnoticed.

4. If they played with sums of three squares, repeat steps 1 through 3 with that game too. Here, the recommendation is to first cross out all the numbers that could be written as sum of two squares, but not as sum of three. This is because those numbers can technically be represented as sum of three squares, but the representation involves 0. Once those are crossed out, the list should be 1, 4, 7, 15, 16, 23, 28, 31, 39, and 47. At this point, students might notice that most some of the square numbers themselves are on the list (1, 4, 16). The rest of the numbers (excluding 28) are an arithmetic sequence that starts at 7 and then increases by 8. If this hypothesis comes up, you can have them test it by verifying that the next couple numbers on the sequence (55 and 63) cannot be represented as the sum of

three squares either. One good conclusion they could reach is that the list of numbers that cannot be expressed as sum of three squares is made up of several sequences.

5. If they played with four squares, after comparing their results, the class should be able to conclude that all numbers can be represented as a sum of four squares.
6. Finally, explain that this is a problem that has interested mathematicians for a long time and that, in her thesis, Dr. Svager described the patterns that show up in the numbers that can be written as sums of two, three, and four squares.
7. To wrap up, motivate a discussion on the game experience. Let students start by sharing about any aspect of it they want. If they need more specific directions, you can ask what they liked about the game and what challenges they encountered. If they played the variations, ask which one they think was the easiest.

Variation: As a last point or assignment, you can ask them to reflect upon what sort of problem would like to study if they had to write a math dissertation.

8. Answer any last questions students may have.

References

- Frazier, Thyrsa Anne. The representation of integers as sums of two, three and four squares. M.A. Thesis. The Ohio State University, 1952.
- Sloane, N. J. A. (ed.). "Sequence A001481. Numbers that are the sum of 2 squares". *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation. Available at https://oeis.org/A001481
- ------------ "Sequence A004215. Numbers that are the sum of 4 but no fewer nonzero squares". *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation. Available at https://oeis.org/A004215

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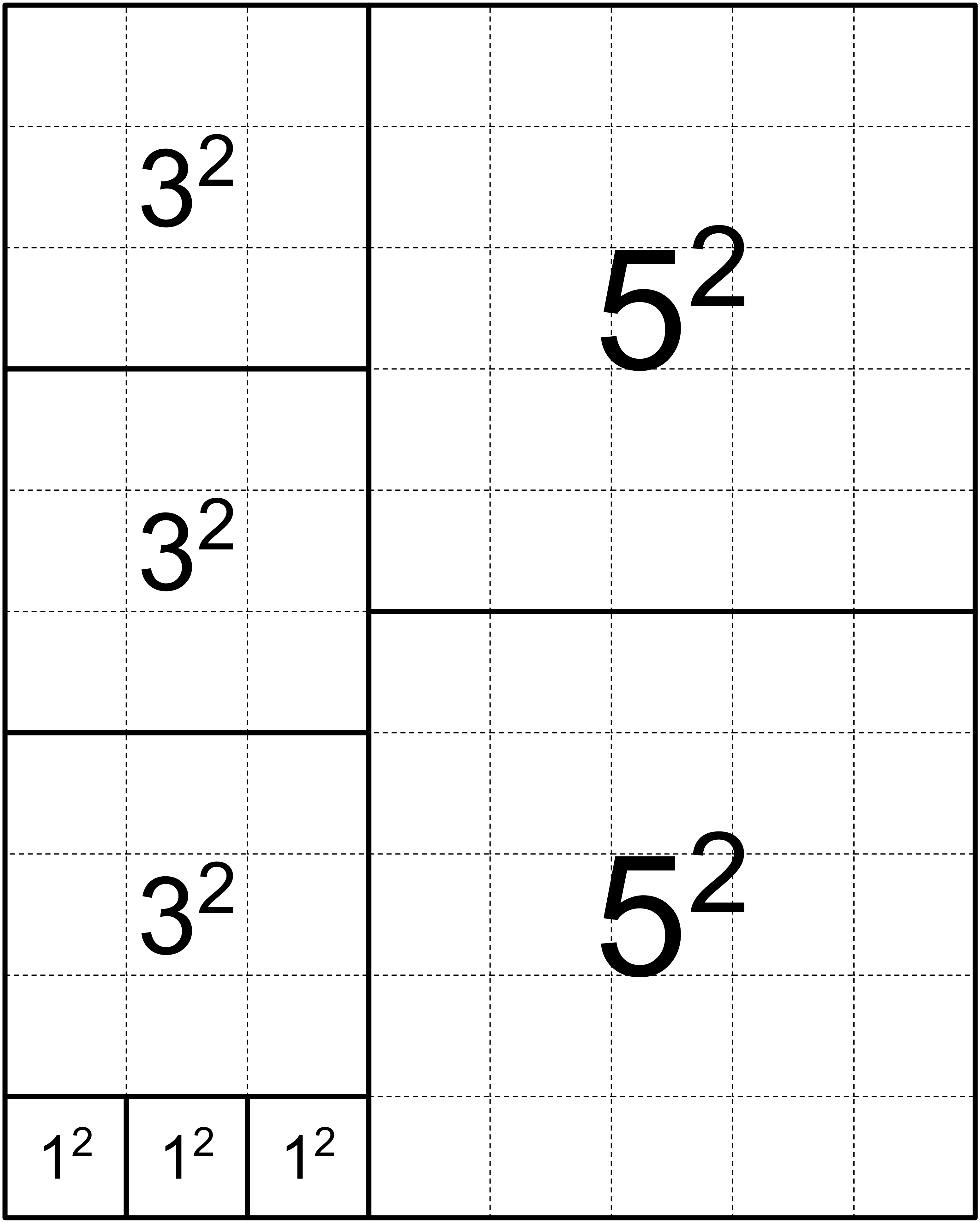
GAME SET

Instructions: The following pages make one game set, except for the last page, which makes materials for four sets. Cut along the thick continuous lines only.

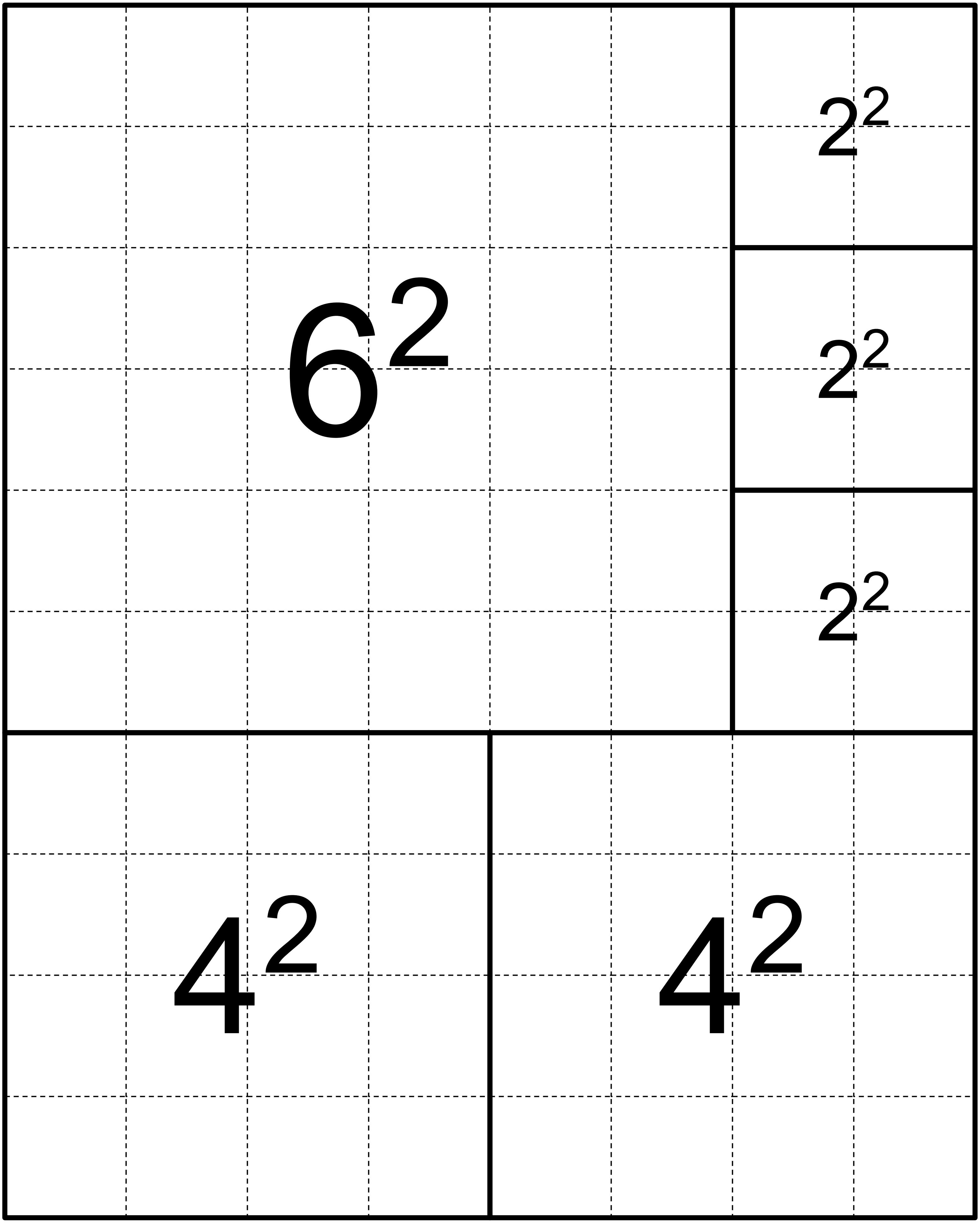
CONTENTS	
Material	Quantity
Number cards	1 of each number (from 1 to 50)
Square card 1	4
Square card 2	4
Square card 4	4
Square card 9	3
Square card 16	2
Square card 25	1
Square card 49	1

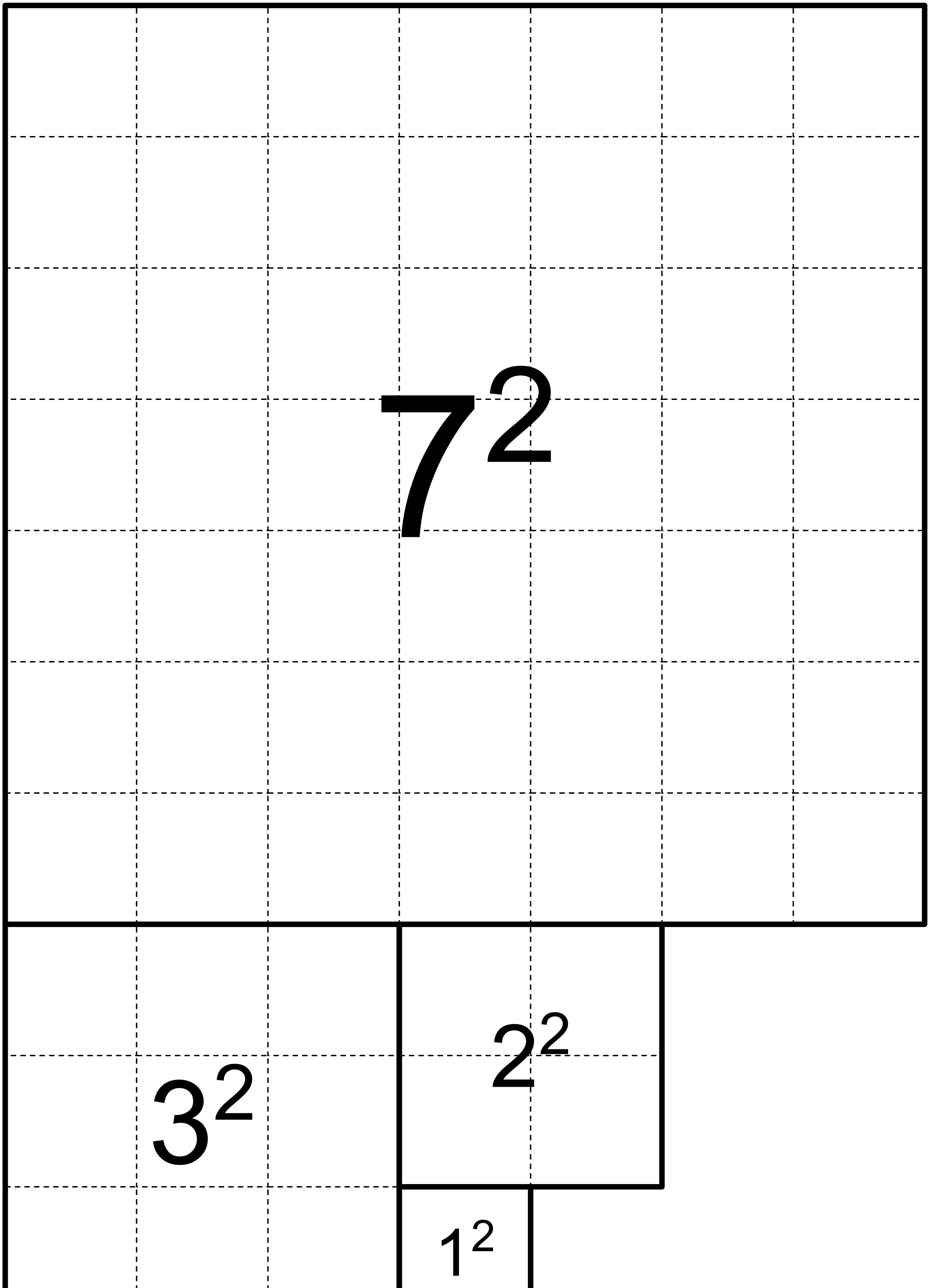
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	29

30	31	32	33	34
35	36	37	38	39
40	41	42	43	44
45	46	47	48	49
50	1	2	3	4



3^2
3^2
3^2
1^2 1^2 1^2
5^2
5^2


6²
2²
2²
2²
4²
4²



7^2
3^2
2^2
1^2

4^2	4^2
4^2	4^2

Note: Make only one copy of this page for every four game sets.