THE OHIO STATE UNIVERSITY

BLACK MATH HISTORY
HIDDEN FIGURES REVEALED

Sum of Squares Game

Grade or age level: Grades 8 – 12 **Time:** 70 minutes **Form of work:** Pairs

Background

Thyrsa Svager is one of the seven Black mathematicians that have obtained a PhD in mathematics at The Ohio State University. On her master's thesis, she explored the problem of describing the numbers that can be represented as the sum of two, three or four squares. Here, the word "squares" refers to square numbers or perfect squares, that is numbers that are the resulting product of an integer with itself.
In this activity, students explore some of the concepts on Dr. Svager's thesis through a two-player board game. An alternative, simpler game is offered for younger students.

Sum of two squares

To check if an integer can be represented as the sum of two squares, one has to look at its prime factorization. It should not contain any primes that are 1 less than a multiple of 4, unless said prime appears an even number of times. Formally, an integer greater than one can be written as a sum of two squares if and only if its prime decomposition contains no factor p^k, where $p \equiv 3$ (mod 4) and k is odd.
Examples:

- $6 = 2 \times 3$ cannot be written as the sum of two squares because its prime factorization contains a single 3 (and 3 is 4 – 1).
- $18 = 2 \times 3 \times 3$ can be represented as sums of two squares, because even though 3 is 1 less than a multiple of 4, it appears two times.
- $130 = 2 \times 5 \times 13$ can also be represented as sums of two squares because it does not contain any primes that are 1 less than a multiple of 4 (both 5 and 13 are 1 more than a multiple of 4, 5 = 4 + 1 and 13 = 12 + 1).

Sum of three and four squares

The only case where a number cannot be written as the sum of three squares is when it's the product of a power of 4 times a number that is one less than a multiple of 8. Formally, the Legendre's three-square theorem states that the only numbers that cannot be written as sum of three squares are those of the form $4^a(8b + 7)$ with $a, b \geq 0$.
Examples:

- 4 and 16 can be written as the sum of three squares because, even though they are both powers of 4, they are not multiplied by a number that is one less than a multiple of 8.
- 7 cannot be written as the sum of three squares because 7 is one less than a multiple of 8 and 1 is a power of 4 ($7 = 1 \times 7 = 4^1(8(0) + 7)$).
- $16 \times 7 = 112$ and $4 \times 15 = 60$ are other examples of numbers than cannot be represented as sums of three squares.

Finally, Lagrange's four-square theorem states that all numbers can be represented as the sum of four squares.

Some example representations

Number	Two Squares	Three Squares	Four squares
4	$0^2 + 2^2$	$0^2 + 0^2 + 2^2$	$1^2 + 1^2 + 1^2 + 1^2$
6		$1^2 + 1^2 + 2^2$	$0^2 + 1^2 + 1^2 + 2^2$
7			$1^2 + 1^2 + 1^2 + 2^2$
16	$0^2 + 4^2$	$0^2 + 0^2 + 4^2$	$2^2 + 2^2 + 2^2 + 2^2$
18	$3^2 + 3^2$	$1^2 + 1^2 + 4^2$	$0^2 + 1^2 + 1^2 + 4^2$
21		$1^2 + 2^2 + 4^2$	$0^2 + 1^2 + 2^2 + 4^2$
60			$1^2 + 1^2 + 3^2 + 7^2$
112			$2^2 + 2^2 + 2^2 + 10^2$
130	$7^2 + 9^2$ or $11^2 + 3^2$	$0^2 + 7^2 + 9^2$	$1^2 + 2^2 + 5^2 + 10^2$

If you need to quickly find a two, three, or four squares representation, we recommend using the following website, which is a powerful calculator: https://www.wolframalpha.com/ On the search bar, type the equation you need to solve, namely $a^2 + b^2 + c^2 + d^2 =$ the number you are searching for, using as many squares as you need. Press enter and wait for results to load. It will give more information that you need but look for the "Integer solutions" subtitle to find the representation you need.

Lesson Details

Objectives

- To discover the hidden lives of Black mathematicians from Ohio State.
- To realize that the valuable work and stories of some people remain hidden and we can do something to change that.
- To practice multiplying and adding.
- To work with the concepts of square numbers and prime decomposition.
- To identify patterns.

Links with Standards

SOCIAL SCIENCES	
Grade	Standard
2	4. Biographies can show how peoples' actions have shaped the world in which we live.
4	8. Many technological innovations that originated in Ohio benefitted the United States

MATHEMATICS	
Code	Standard
4.OA.4	Gain familiarity with factors and multiples.
8.EE.1	Understand, explain, and apply the properties of integer exponents to generate equivalent numerical expressions.
Practice 1	Make sense of problems and persevere in solving them.
Practice 2	Reason abstractly and quantitatively.
Practice 3	Construct viable arguments and critique the reasoning of others.
Practice 8	Look for and express regularity in repeated reasoning.

Materials

- Thyrsa Svager poster
- Thyrsa Svager video
- Projector
- One game set per group of four. Each game set includes:
  - One 6x6 board
  - Prime cards deck (72 cards total)
  - Square cards deck (90 cards total)
- Tracking sheet (you can use the one included with the game or they can make their own)
- Pencil

Opening Activities/Motivation (20 min)

1. Introduce Dr. Thyrsa Frazier Svager using the slides that you can access on this link. The set of slides contain the following short biography of her and a series of question to reflect upon her life and circumstances. Discuss the questions with the class.

   Born in 1930 in Wilberforce, Ohio, Thyrsa Frazier Svager achieved what few African-American women of her generation have in the field of education. She was one of the first African-American women in the United States to earn a Ph.D. in mathematics. Her mother (a professor) and her father (a statistician) instilled the importance of education in her from a young age. Thyrsa Svager graduated from the Wilberforce University Preparatory Academy at the age of 15, going on to complete her undergraduate studies at Antioch College in Yellow Springs, and earning both her master's degree and doctorate in mathematics at The Ohio State University. Dr. Svager worked as a statistical analyst at Wright-Patterson Air Force Base and as an instructor at Texas Southern University in Houston. She spent most of her career at Central State University, where she not only served as professor, but also as administrator, Dean, Provost, and Interim President.

   - Dr. Svager was 1 of 4 black students at her college at the time of enrollment. How do you imagine she felt? What would you have done if you had been her classmate?
   - Do you think her parents' professions might have influenced her career choice?
   - What are your parents' occupations? Would you like to work on the same fields?

2. Introduce the activity by mentioning that the game they are going to play is inspired by Svager's master thesis work "The representation of integers as sums of two, three or four squares." Ask students what the word "squares" means in this context. If they only give answers referring to the polygon, clarify that the title really means "square numbers" or "perfect squares" and then invite them to share what they know about those numbers. Make sure to include some examples and counter examples in the discussion.
3. Continue explaining the title of the thesis, now focusing on the part about the sum. Svager was studying three related problems: numbers that result from adding two squares, numbers that result from adding three squares, and numbers that result from adding four squares. For example, $1^2 + 2^2 = 1 + 4 = 5$. Ask a couple students to give other examples. Not all numbers can be written as the sum of two squares, as they will see in the game. Give an example of this, like, 3 which can only be broken up into the sum of two non-negative integers as 3 = 1 + 2 or 3 = 0 + 3, but neither 2 nor 3 are square numbers, so it doesn't work. However, 3 can in fact be written as the sum of three squares: $3 = 1^2 + 1^2 + 1^2$. After giving that example, it is worth pointing out that 0 is a square number, since $0^2 = 0$. Challenge the students to give a few examples of sums of three and four squares, as well as numbers that cannot be represented as the sum of two or three squares.
4. Finally, mention that master theses are often an exhaustive review of previous work done by others in a specific field. In the case of Svager, she chose to survey this topic, but the problems she analyzes were solved before the 19th century.

Procedures (30 min)

Game Components

- Game board. Simply a 6-by-6 grid.
- Prime cards. Cards that have a prime number on them (such as 3, 5, or 7). They are placed on the board and multiply with each other in a straight line (horizontally or vertically).
- Square cards. Cards that have a square number on them (such as 2^2, 5^2, or 8^2).
- Goal. Players use square cards on their hands to add up to the products on the board, in which case the cards that make up the product are flipped. The game is over when the board is full and no more prime cards can be flipped. The winner is the player that used the most square cards.

Game Rules

- This is a two-player game. However, for educational purposes, we recommend students play two against two.
- Prime cards and square cards are mixed separately, and each player is dealt four prime cards and four square cards. The rest of the cards are placed on the table in two face-down piles.
- The players cards are shown face up on the table, always visible to both players.

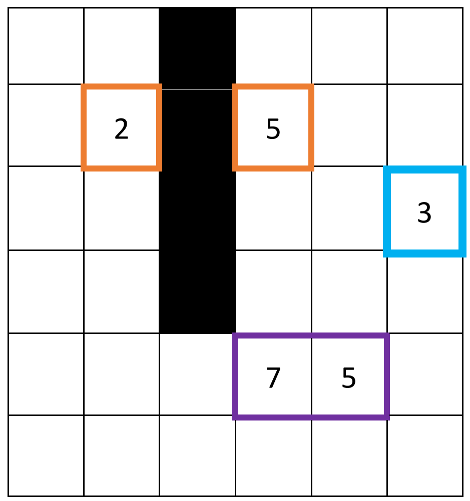
- The player with the largest product of prime cards goes first. If there is a tie, the player with the largest sum of square cards goes first. If there is another tie, decide arbitrarily.
- On a player's turn, they must place a prime card on any empty tile of the board.
- Additionally, the player on turn can claim the tiles of a current product on the board if they have at least two square cards that sum to that product. Note that all the contiguous cards on a given row or column must be taken for the product.
- To claim tiles, flip the cards on them so they are face down. The player should also keep the square cards they used.
- A player may claim numbers before or after they place a prime on the board (which is required each turn), or they can choose not to claim at all during their turn.
- A player may also claim multiple numbers in one turn. If a player attempts to incorrectly claim a number, nothing happens (the board and the player's cards are left unchanged).
- Claimed tiles act as barriers on the board. They no longer multiply with neighboring cards and players can no longer place primes on the claimed tiles.
- To make arithmetic easier, we suggest keeping track of the row and column products as new number cards are placed onto the board and as numbers are claimed.
- At the end of a turn, the player must draw one prime card and one square card. In addition, if the player claimed any products, they must also draw one square card for every product claimed (regardless of how many square cards they used to claim the number).
- The game is over when the board is full and no more tiles can be claimed.
- The player with the most used square cards wins.

Example Game

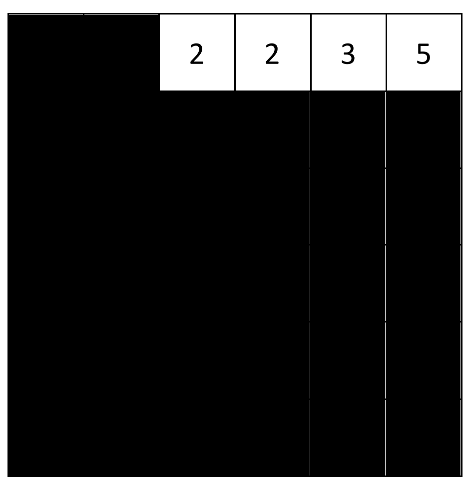
The products on the following board are $2 \times 3 \times 5 \times 3 = 90$ (third column from the left with four contiguous prime cards), $2 \times 3 \times 5 = 30$ (second row from the top, with three contiguous prime cards), $7 \times 5 = 35$ (second row from the bottom, with two prime cards), and 3(single prime card with no neighbors).

		2			
	2	3	5		
		5			3
		3			
			7	5	

If the player in turn has the square cards 9^2 and 3^2, they can claim the tiles on the third column (green) since $90 = 81 + 9 = 9^2 + 3^2$. Such move would score 8 points, since they claimed 4 prime cards using 2 square cards (the score is the product of those two numbers).
Once those tiles are claimed, the orange number (second row) is now broken up into two single card numbers. 2 and 5 no longer can multiply each other. However, other cards can be added to the left of 2 and to the right of 5.



If both players only have two squares left, the following board shows a finished game since the number cannot be written as a sum of two squares (3 = 3 mod 4 and the number has an odd number of 3s):



In the classroom

1. Start by explaining the rules of the game. Show the examples to clarify the instructions. Give the students a chance to ask questions until the group understands the process.
2. If the group is not familiar with the concept of prime numbers, dedicate some time to briefly explain it.
3. If necessary, you can play a few turns against the whole group on a screen to make rule rules have been understood.

4. Remind the students to write down the products and the squares they use to sum up to those. They can use the tracking sheet in the Game Set section of this document or they can make their own tracking system on a piece of paper. This will be important in the closure to discuss what numbers can and which can't be written as sum of squares.
5. Split the group into groups of fours, to play one pair against another pair, and distribute the materials. Although it is also possible to play one-against-one, we recommend the four-player format so moves can be discussed. We find that it is helpful to share the thought process and to help each other with the arithmetic.
6. Have them play one round. Go around the room checking that everybody understood the rules correctly.
7. If some pairs finish faster than others, allow them to play a second round.
8. Once every pair has played at least one full round, have the group clean up and go back to their sits for discussion. They should have their sheets with products and sums of squares handy.

Closure (20 min)

1. Motivate a discussion on the game experience. Let students start by sharing about any aspect of it they want. If they need more specific directions, you can ask what they liked about the game and what challenges they encountered.
2. Next, ask about strategies, whether they found any that allowed them to make better moves (claim more tiles, block the opponent, what factor cards is it better to play).
3. Now start a discussion about prime factorization. Start by asking what numbers could result from the products on the board. The goal is to notice that, even though every number has a (unique) prime decomposition, some numbers could not result on the board simply because the game only includes a few primes. For example, since there's no card for 17, then any multiple of 17 could never result as a product on the board.
4. At this point, continue with the discussion asking if they noticed any patterns in the products they were able to claim and those that they couldn't.
5. Write on the board some examples of the products they claimed as sum of two squares and ask again if they see any patterns, specifically on the products. Whether the patterns are evident or not will depend on the specific examples that came up for them on the game, but hopefully you'll be able to point out at least a couple. If patterns do not come up at all, you can reveal that, as Dr. Svager described on her thesis:
   - When a number is written as product of primes and there is a prime like 3, 7, 19, and 23, then it is impossible to find two squares that add up to that number. This is unless that prime repeats an even number of times. Depending on the level of your group, you can further explain what is the common characteristic of those primes (the are all one less than a multiple of 4).
   - All numbers can be written as the sum of four squares.
6. To note that last point, challenge them to find four squares that add up to their birthday day number.

7. As a last point, ask them to reflect upon this: if they had to write a math thesis, what sort of problem would they like to write it about?
8. Finally, answer any last questions they might have.

References

- Frazier, Thyrsa Anne. The representation of integers as sums of two, three and four squares. M.A. Thesis. The Ohio State University, 1952.
- Sloane, N. J. A. (ed.). "Sequence A001481. Numbers that are the sum of 2 squares". *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation. Available at https://oeis.org/A001481
- ------- "Sequence A004215. Numbers that are the sum of 4 but no fewer nonzero squares". *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation. Available at https://oeis.org/A004215

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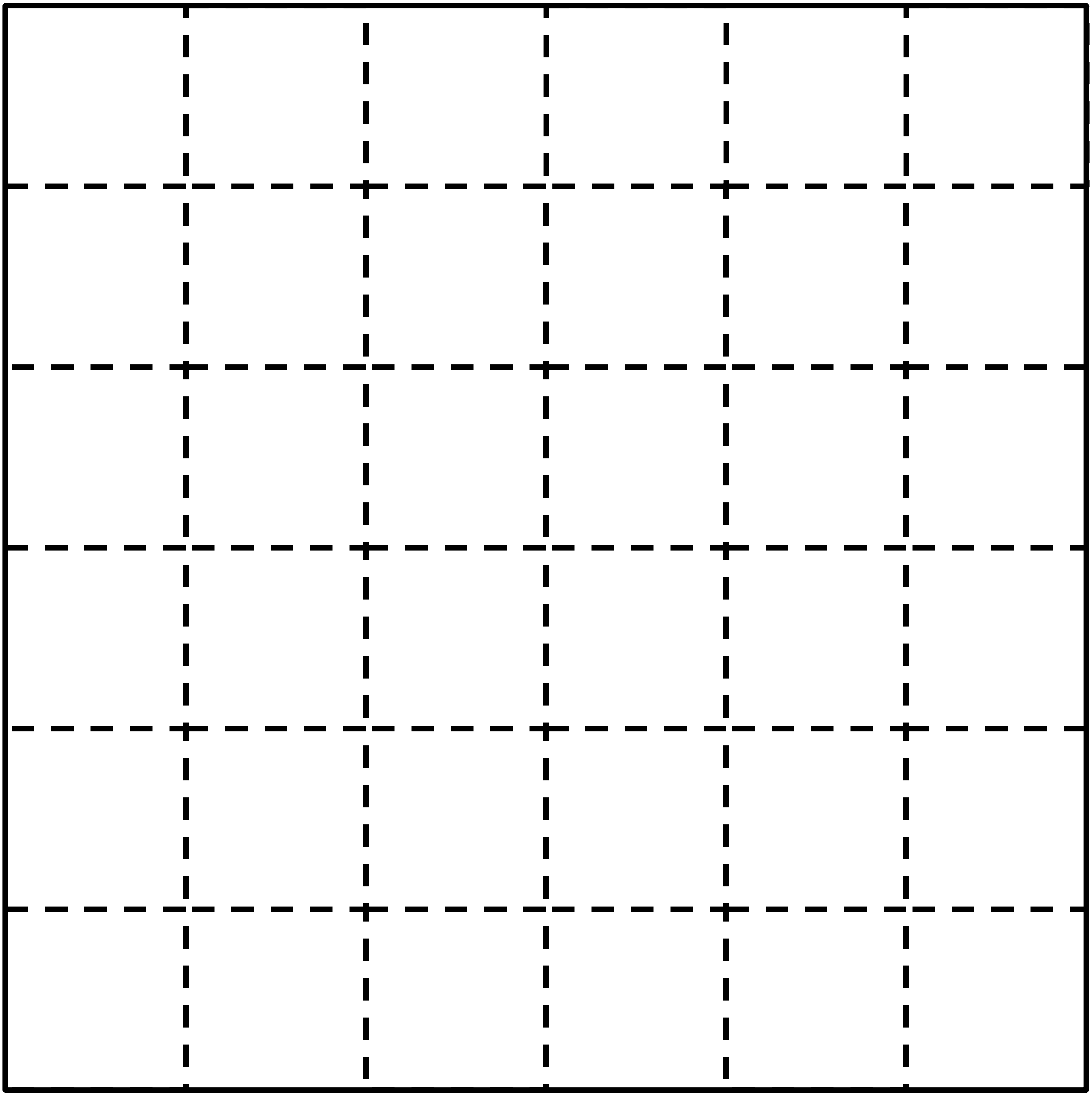
GAME SET

Instructions: The following pages make one game set. The grid on the next page is the board. Cut along the thick continuous lines only.

CONTENTS	
Item	Quantity
6x6 grid	1
Prime card (2)	16
Prime card (3)	8
Prime card (5)	16
Prime card (7)	8
Square card (0^2)	6
Square card (1^2)	6
Square card (2^2)	6
Square card (3^2)	6
Square card (4^2)	4
Square card (5^2)	4
Square card (6^2)	4
Square card (7^2)	3
Square card (8^2)	3
Square card (9^2)	3
Square card (10^2)	3
Square card (11^2)	3
Square card (12^2)	3
Wild square card ($?^2$)	6



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TRACKING SHEET

Contiguous prime cards on a row or column	Product of those primes	Possible sums of squares that result in that product

2	2	3	5	5	7
2	2	3	5	5	7
2	2	3	5	5	7
2	2	3	5	5	7
2	2	3	5	5	7
2	2	3	5	5	7
2	2	3	5	5	7
2	2	3	5	5	7

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$
$4^2 = 16$	$5^2 = 25$	$6^2 = 36$
$7^2 = 49$	$8^2 = 64$	$9^2 = 81$
$10^2 = 100$	$11^2 = 121$	$12^2 = 144$

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$
$4^2 = 16$	$5^2 = 25$	$6^2 = 36$
$7^2 = 49$	$8^2 = 64$	$9^2 = 81$
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$1^2 = 1$	$2^2 = 4$	$3^2 = 9$
$4^2 = 16$	$5^2 = 25$	$6^2 = 36$
$0^2 = 0$	$0^2 = 0$	$0^2 = 0$

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$
$0^2 = 0$	$0^2 = 0$	$0^2 = 0$
$?^2$	$?^2$	$?^2$
$?^2$	$?^2$	$?^2$