Dragon Fractal

Grade or age level: Grades 4- 8 **Estimated Time:** 75 minutes **Form of Work:** Individual

Background

Dr. William McWorter is one of the seven Black mathematicians to have earned a PhD in mathematics from The Ohio State University. He also obtained bachelor and master's degrees from OSU. In his career, Dr. McWorter studied group theory, subgroups, and finite sequences. His thesis was written on combinatorial mathematics. Additionally, Dr. McWorter published research on fractals and how they might bring order to chaos. More specifically, he explored dragon fractals using computer algorithms but also folding paper, thus creating new and unique fractal formations.

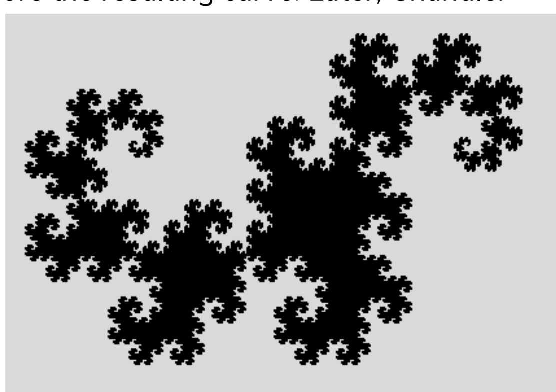
Fractals

These mathematical objects are well known in popular culture. Loosely speaking, a fractal has two distinguishing characteristics: they are self-similar and they are constructed through a recursive method. The self-similarity property means that if you zoom in on a specific part of the fractal, you would see the same picture as before zooming. A recursive method is one where the same steps are repeated over and over with the resulting object.

Some well-known examples of fractals include Sierpinksi's triangle, which is used in modern-day motion capturing in famous movies such as *Spider-Man: Far From Home*.

What are dragon fractals?

The Dragon curve or fractal was invented by a NASA physicist, John Heighway, in 1966. He was thinking about the pattern that would emerge from repeatedly folding a $1 bill. Heighway and a couple of colleagues proceeded to further explore the resulting curve. Later, Chandler Davis and Donald Knuth developed the theory. Nowadays, the term "dragon fractal" refers to a whole category of fractal curves.



Dragon fractals were popularized in the *Jurassic Park* book by Michael Crichton.

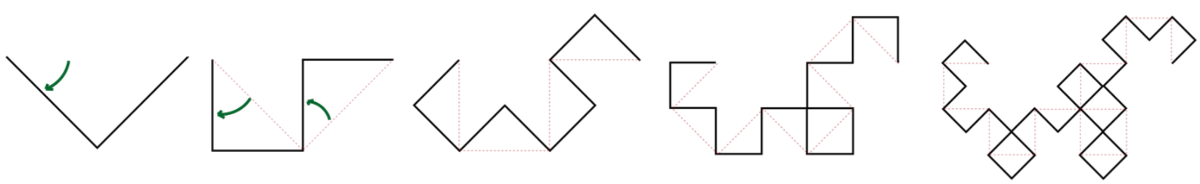
As previously mentioned, the original Dragon curve can be created by recursively folding a strip of paper in half, always in the same direction. To get the curve, one must unfold the strip of paper and arrange it so that every fold makes a 90° angle. Of course, a real piece of paper can be folded only so many times. The actual Dragon Fractal is what would form after an infinite number of folds – if that were possible. To explore the curve beyond what folding paper allows, Heighway and his colleagues drew the subsequent curves on graph paper.

Dragon fractals grow into more complex shapes day-by-day. Note that the term "day" is arbitrary and just refers to the stage in the recursive process. Each day, the unit develops a 90° fold along the midpoint of the endpoints between folds. For instance, on day two, there should be one 90° fold, but on day three, we have three 90° folds and four segments. This is because we developed one fold on each side of the already-existing fold for a total of three folds.

The mathematicians who studied these patterns discovered several other ways of building the curve, which we describe next.

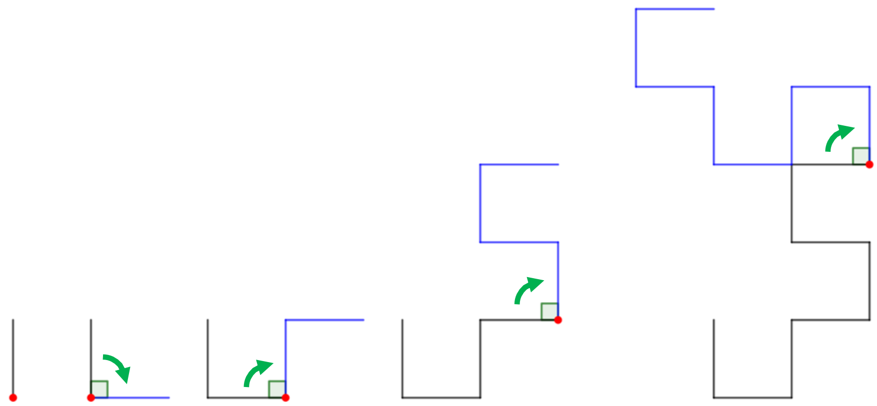
Geometric construction

Start with a segment (day 1). Now imagine that segment is the hypothenuse of an isosceles right triangle. On the second day, draw the legs of that triangle and erase the hypothenuse. On the following days, do the same process with each segment. Alternate building the triangle towards the outside and the inside of the curve.



Another geometric construction

Another geometric construction of the dragon goes as follows. Again, start with a segment. To make this explanation easier to follow, let's imagine that it is a vertical segment and that the upper end is going to be our fixed starting point, while on the bottom end we imagine we have a hinge. For the second day, we develop a copy of the original segment and move it on the hinge 90° in a clockwise direction. We then imagine that the new end point of the curve has a hinge on it and repeat the process for the third day. The construction continues in this manner. In the following image, the red points represent the hinges, and the blue lines are the copies.



Number construction

After unfolding the strip of paper, one can pay attention to the sequence formed by peaks and valleys (basically, folds with the vertex pointing up or folds with the vertex pointing down). In this manner, the state of the dragon at each day can be represented by a sequence of letters P for the peaks and V for the valleys or, to simplify, use 0 to represent peaks and 1 to represent valleys.

Note that the first day as it is depicted on the first geometric construction above would be a valley. However, it could also be seen as a peak if you turn the paper upside down. It doesn't matter if you choose to start this way, what matters is to be consistent and look at the strip always from the same perspective.

The important part of this number sequence is that, once you've noticed the pattern, it can be written without folding a strip of paper or looking at the geometrical constructions.

This is the recipe for writing the number that represents the following day:

- Write the number that corresponds to the previous day twice and put a 1 in between the two instances.
- On the second instance (the one to the right), replace the middle digit by the opposite (so 1 should change to 0, and 0 should change to 1).

Here's how the first few days go:

1 -> 1 1 0 -> 110 1 100 -> 1101100 1 1100100

The digit highlighted in blue is the 1 that is put in the middle, as explained above. The red digit corresponds to the middle digit of the second instance, which is replaced by the opposite.

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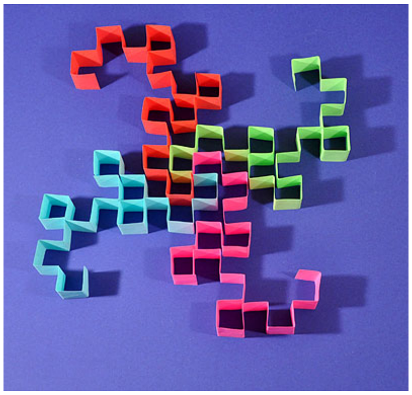
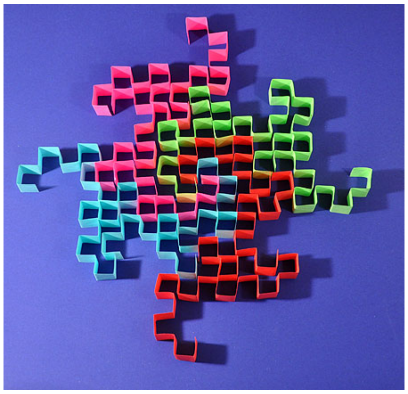
Dragon Tiles

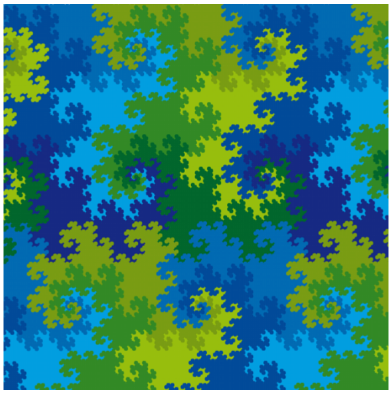
Another amazing property of the Dragon Fractal is that it can make tessellations. In other words, several Dragon Curves can be used as tiles, because they fit nicely with each other. The following pictures show how the paper dragons fit with each other, but also how the Dragon Fractal, the limit curve after an infinite number of days, tiles the plane.



5-day dragons



6-day dragons



Infinite-days dragons

Objectives

- To discover the hidden lives of Black mathematicians from Ohio State.
- To realize that the valuable work and stories of some people remain hidden and we have the power to rectify this.
- To introduce recursive algorithms.
- To recognize complex patterns and identify variations.
- To introduce or solidify understanding of infinity.

Links with Standards

SOCIAL SCIENCES	
Grade	Standard
2	4. Biographies can show how peoples' actions have shaped the world in which we live.
4	8. Many technological innovations that originated in Ohio benefitted the United States

MATHEMATICS	
Code	Standard
4.OA.5	Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.
Practice 1	Make sense of problems and persevere in solving them.
Practice 2	Reason abstractly and quantitatively.
Practice 3	Construct viable arguments and critique the reasoning of others.
Practice 8	Look for and express regularity in repeated reasoning.

Materials

- William McWorter poster
- Construction or scrap paper cut into 1-inch strips (a variety of colors is recommended for student engagement and interest but not necessary), at least two per student.
- 1 1/8 inches graph paper (printable template included)
- Scissors
- Rulers
- Tape
- Pencils
- (Optional) Colored pencils, crayons, markers, etc.
- (Optional) Hairpins

Opening and Motivation (25 minutes)

1. Introduce Dr. William McWorter and his studies to students using materials gathered from the Hidden Figures Revealed project.
2. Explain to students that Dr. William McWorter, among other fields of study, studied fractals. Ask students about their previous experiences with the word, if they have heard of fractals before. Introduce the idea of fractals, which are "created by repeating a simple process over and over in an ongoing feedback loop" (What are Fractals?). In this lesson, students will be exploring a specific fractal called a dragon fractal by folding a piece of paper over and over again.
3. Stress to students that math does not have to be hard to understand and that anybody, at any skill level, can explore math and, in doing so, call themselves a mathematician. Dr.

William McWorter, who has a PhD in mathematics, deepened his higher-level understanding of mathematics by folding paper, just like students will.

Procedures (35 minutes)

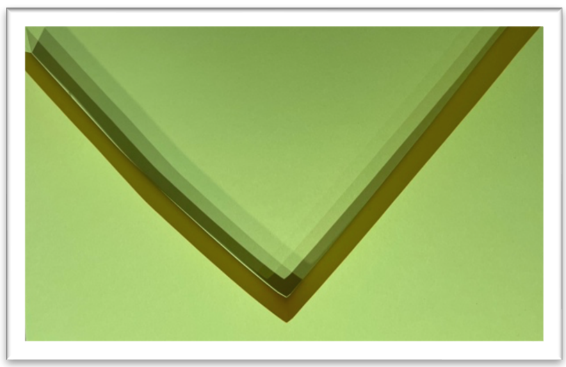
1. Distribute worksheets to each student.
2. Grab at least one piece of construction paper.

Variations: Using varied colors is recommended, but this activity could also be completed with scrap paper or normal printer paper. For a chance to add some creativity and develop artistic skills, students can also color the paper prior to cutting in and see their designs be represented in the final product.

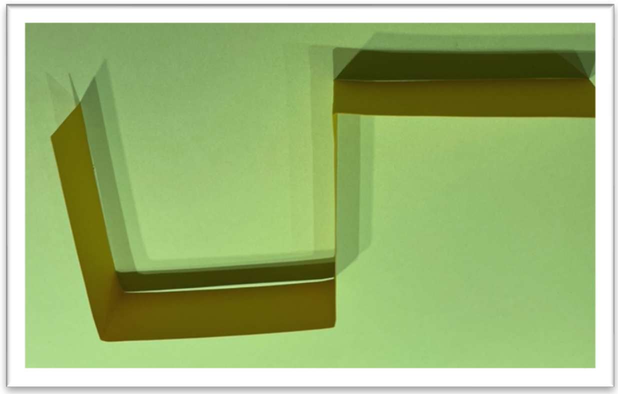
3. Ask students to use their rulers to mark the short end of their paper in inch-long increments. Have them cut their paper into strips.

Note: If teachers pre-cut their strips using paper cutters and have students collect strips from a pile/bin to save on time, skip steps 1-2.

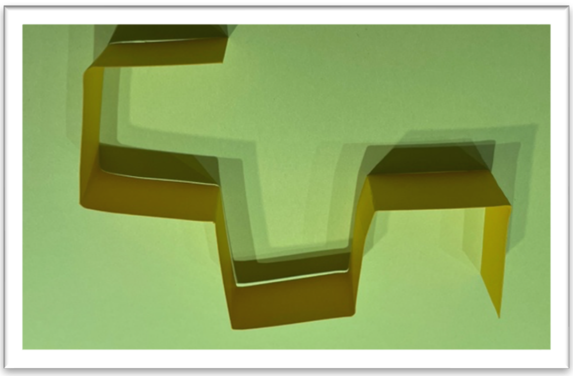
4. We recommend taping two strips of paper together (on the short end) to make it longer. This will allow for easier manipulation.
5. Fold the piece in half. Unfold the paper and have students record their dragon. This is “day 2” of their dragon. Remind students to record their dragon’s number representation on their worksheet after each step.



6. Fold the piece in half again. Make sure the folded edge is to the right and that they fold the paper such that the folded edge is now on the left. Ask students to unfold and record their dragon’s “code” on their worksheets.



7. Repeat this process two more times. At this point, it would be difficult to fold their strip any further.



8. Tell students to grab more strips of paper (a different color would yield a more interesting result, but not necessary). Have them repeat steps 4 to 7 (this time it is not necessary that they draw the process of the dragon). Once finished with their additional dragon, have them tape their dragons together. Ideally, they should be taped forming a 90° angle on the ends, but at this point, you can give students more creative liberty.

9. When students complete both their worksheet and their dragon fractals, have them compare their results with the students around them. Motivate students to discuss both the physical representation of their dragon and the number representation.

Variation: With older students you can direct them to tape the two dragons in the correct way so that they form the day 5 dragon. Use the provided template as aid.

Variation: This variation can follow the previous one, but it can also be done if the previous one was not. Also with older students, you can make a class art project where they all put their dragons together to form a tessellation. Depending on the characteristics of your group, you could also do this by splitting the class in two or three big groups. Once all the dragons are aligned in the same position, the tessellation is not so hard to build. All you need to do is rotate each dragon 90° with respect to the previous one, always moving in the same direction. Four dragon heads (or tails, depending on how you see it) meet at each corner.

Closure (15 minutes)

Ask your students to share their thoughts about the experience. You can explain that since the paper construction gets tedious after a while, mathematicians looked for patterns on the dragon's growth so that they could predict what the next day dragon would look like without having to build it on paper. With this, they were also able to tell computers how to draw dragons for many days.

Optional: Draw the first two days of the dragon and challenge the class to draw the third day without looking at their dragons or their graph paper. If they recall the pattern noticed during the lesson, they should be able to draw the next steps following the algorithm described under "Another geometrical construction" in the Background section.

Motivating discussion questions:

1. Why do you think this is called a dragon curve?

   Answer: One of the first people who study it gave it that name because he thought it resembled a dragon.

2. Have students look at the complex dragon fractal on the next page and ask them why it is so much more complex than their paper dragons. Do they think they could make one that big on paper?

   Answer: This dragon went through many more "days." It is technically possible to build one like that with paper, but it takes a lot of work.

3. Ask students to provide their guesses for how many "days" old is this dragon. Encourage students to do work, make educated guesses.

   Answer: It is a day 10 dragon.

4. Look at the number representations each student generated based on their dragon. Ask them what patterns did they notice?
5. Ask students why they think Dr. McWorter was interested in learning and exploring fractals.

   Possible answers: He was interested in algorithms and fractals like the dragon curve are interesting because they follow very simple recursive algorithms to give life to something complex and beautiful.
6. Motivate students to think about other folding patterns they could use to create their own piece.
7. Finally, ask students who can be a mathematician? What does it mean to be a mathematician? Where do math problems come from? Does analyzing patterns in folding paper make you a mathematician? (yes!)

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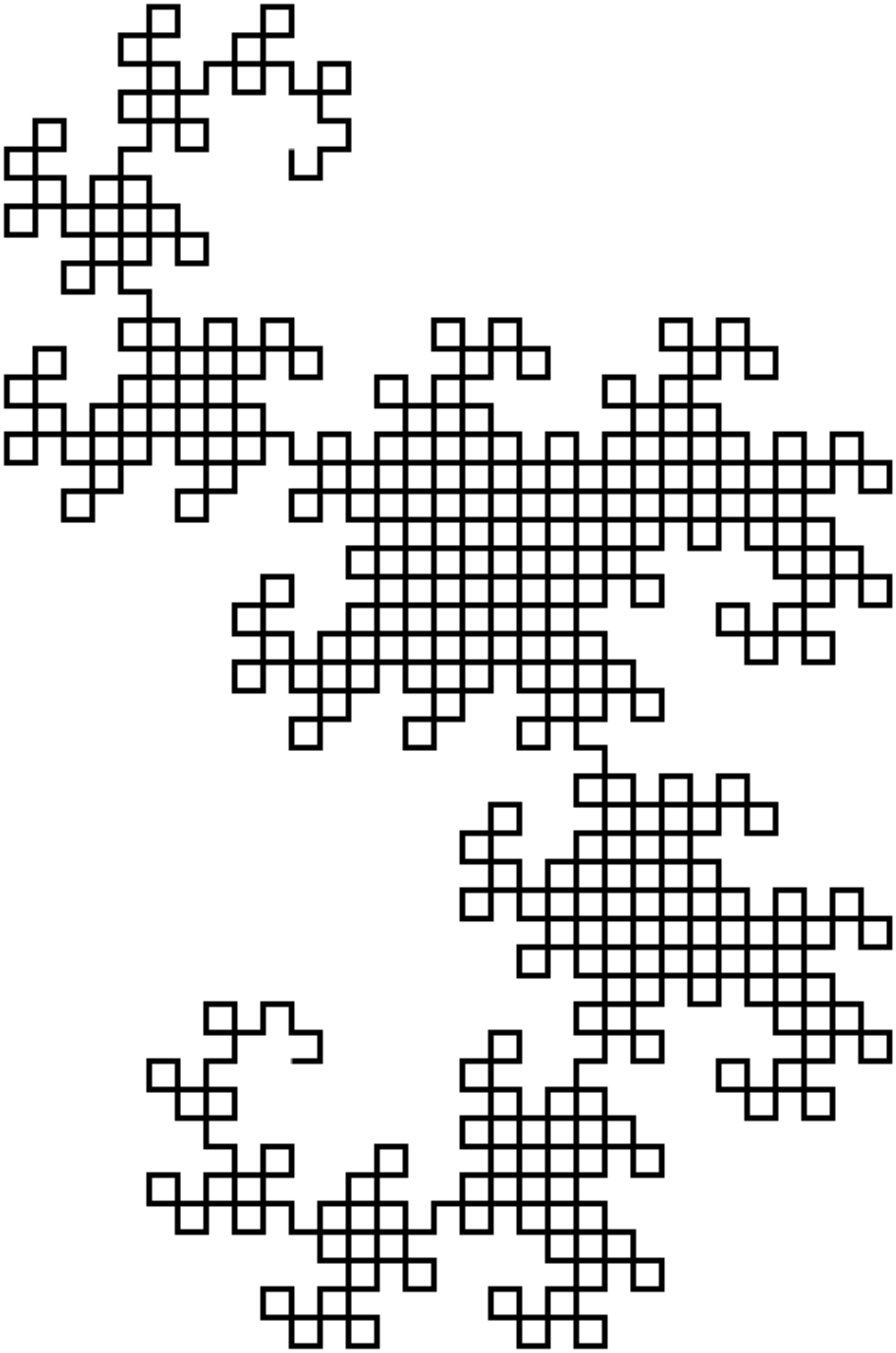

References

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  - This video contains an activity educators can use to build upon the knowledge gained in this activity or as an alternative option.
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